LEVEL

| REPORT DOCUMENTATION PAGE | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|

| 1. REPORT NUMBER | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
|---|---|---|
| 14092.20-M | AD-A104 293 | |

| 4. TITLE (and Subtitle) | 5. TYPE OF REPORT & PERIOD COVERED |
|---|---|
| Boundary Observability and Controllability for Diffusion Equations. | Final Report, 10 Dec 76 — 30 Jun 80 |
| | 6. PERFORMING ORG. REPORT NUMBER |

| 7. AUTHOR(s) | 8. CONTRACT OR GRANT NUMBER(s) |
|---|---|
| Thomas I. Seidman | DAAG29-77-G 0061 |

| 9. PERFORMING ORGANIZATION NAME AND ADDRESS | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
|---|---|
| University of Maryland Baltimore, MD 21228 | |

| 11. CONTROLLING OFFICE NAME AND ADDRESS | 12. REPORT DATE |
|---|---|
| U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709 | Sep 81 |
| | 13. NUMBER OF PAGES |
| | 14 |

| 14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office) | 15. SECURITY CLASS. (of this report) |
|---|---|
| | Unclassified |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

**16. DISTRIBUTION STATEMENT (of this Report)**

Approved for public release; distribution unlimited.

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**17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)**

NA

**18. SUPPLEMENTARY NOTES**

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**19. KEY WORDS (Continue on reverse side if necessary and identify by block number)**

**20. ABSTRACT (Continue on reverse side if necessary and identify by block number)**

Investigation has focused on problems of observation and control for parabolic equations, primarily with interaction limited to the boundary of the region. Abstracts and bibliography presented herein may be classified under six headings: A. Basic theoretical questions on control of parabolic equations; B. Problems of observation and identification for parabolic equations; C. Other system-theoretic problems; D. Problems and considerations (algorithms, regularity, etc.) related to approximation and numerical computation; E. Ill-posed problems; and F. Problems involving nonlinear partial differential equations.

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FINAL REPORT

ARO grant DAAG-29-77-G-0061

Thomas I. Seidman,
Principal Investigator
University of Maryland Baltimore County

"Boundary observability and controllability for diffusion equations"

a. Foreword

b. Problem statements

c. Bibliographical list

d. Summaries of results

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This grant, as extended, covered a four year period (essentially 1977-1980).
In addition, as the principal investigator was out of the country (visiting at
the Université de Nice for the academic year 1980- '81), preparation of this
final report has been somewhat delayed; since much of the work of this additional
half-year is based on and is a continuation of work done during the grant period,
the report actually includes work of 4½ years.  The bibliographical list presents
items in order of publication or expected publication, [1] - [34], with some
brief annotation.  The summary of results following that is in the form of
abstracts, prepared for this report.

During the grant period a total of 21 papers and reports (items [2] - [23]
noting the equivalence of [4], [7]) were prepared.  Of these, 10 have already
appeared or have been accepted for publication in various journals while 8 others
(The 3 appearing in the annual Séminaires IRIA have, somewhat arbitrarily, been
counted with these.)  are in conference proceedings.  The remaining three are :
a note [23] submitted but not yet accepted, a NASA report [8] and a book
chapter [22].

Investigation has focused  on problems of observation and control for para-
bolic equations, primarily with interaction limited to the boundary of the
region.  With considerable overlap, the items [1] - [34] may be classified under
six headings:

A.  Basic theoretical questions on control of parabolic equations.

   [1], [4], [5], [13], [14], [19], [21], [23], [24], [25], [26], [34].

B.  Problems of observation and identification for parabolic equations.

   [1], [2], [3] [5], [6], [11], [18], [22], [27], [33], [34].

C.  Other system-theoretic problems.

   [2], [4], [20], [32].

D.  Problems and considerations (algorithms, regularity, etc.) related to approx-
   imation and numerical computation.

   [3], [5], [6], [11], [12], [13], [14], [16], [18], [19], [22], [26].

E.  Ill-posed problems.

   [6], [8], [9], [11], [16], [17], [18], [22], [27].

F.  Problems involving nonlinear partial differential equations.

   [10], [11], [12], [14], [15], [24], [27], [28], [29], [30], [31], [33], [34].

As general observations, note the following:

(i)   The original proposal focused on basic theoretical questions of observability and controllability.  Much of the published work is relatively abstract/ theoretical in style.  Nevertheless, this work has been motivated by the possibility of application  ([11] is a particular case in point.)  and, in particular, the heading D reflects a concern with considerations relevant to successful numerical computation.  It is this, of course, which.has led to the concerns with regularity of optimal controls and with ill-posed problems.

(ii)   There has been a gradual shift toward greater concern with nonlinear problems.  In part this reflects an attitude that the 'real' problems of application are nonlinear.  The work on linear problems, of course, remains significant - not only as a model but because,(e.g., through linearization) the linear results apply also to consideration of nonlinear problems.

(iii)   At present the most significant outstanding theoretical problem for the linear case remains patch observability/controllability.  From the practical point of view, the outstanding problems are:  (1)  well-posedness of observation in the nonlinear case (for which this is no longer dual to exact null-control ability), (2) improved algorithms for computation of controls and stabilizing feedbacks including cases with constraints, (3) applications related to variational inequalities of evolution including free boundary problems, (4) realistic modeling of stochastic considerations.

b.  Statement of some problems.

Consider a process governed by the parabolic equation:

(1) $$\dot{u} + A u = f$$

where  A  is a second order elliptic operator — say

$$A = -\Delta u \quad \text{or, more generally,} \quad Au = - \nabla \cdot a \nabla u + qu$$

with specification of boundary data as

(2) $$u \quad \text{or} \quad u_\nu = \bar{n} \cdot a \nabla u \quad \text{or} \quad \alpha u_\nu + \beta u \quad \text{on} \quad \partial \Omega .$$

If  $(\alpha u_\nu + \beta u)$  is specified, the complementary data is

$$(-\beta u_\nu + \alpha u) / (\alpha^2 + \beta^2) .$$

Equations of the form (1) govern a wide variety of phenomena associated with diffusion:  heat conduction, diffusion of a solute, etc.

<u>Distributed control</u>:  Select  f  so as to minimize  a 'cost' of the form

(3) $$J = \|f\|^2 + \lambda \|u - \hat{u}\|^2 + \mu \|u(T) - \hat{\omega}\|^2$$

(using, say, appropriate  $L^2$  norms) with  $\hat{u}$  a 'target trajectory' and $\hat{x}$ a 'terminal target' given.  The initial and boundary data are, here, also given.

<u>Boundary control</u>:    Select boundary data  $\varphi$  so as to minimize (3) with $\varphi$ for  f.  In applications this is a typical problem as one often has no possibility of interaction except at the boundary.  Indeed, one's interaction may be limited to a 'patch' on the boundary (i.e., $\Gamma_a$ = 'active' boundary $\subset \partial \Omega$, with the constraint:  $\varphi = 0$  on  $\partial \Omega \backslash \Gamma_a$) and one speaks of 'patch control'.

<u>Boundary observation</u>: The equation and boundary conditions are assumed known (e.g., homogeneous) but the initial state unknown.  Given observation of the complementary boundary data as well (possibly only on  $\Gamma_a$ — patch observation one wishes to use this to determine the state  u(T).

One speaks of (boundary) observability if this last is possible.  It is known — for linear problems (the corresponding problem for nonlinear equations is an interesting target for future investigation) — that this well-posed if and only if one has <u>exact nullcontrollability</u>: for each initial state there is a control such that  u(T) = 0.

In these contexts one seeks information on the existence and characterization of optimal controls, continuity of dependence on various aspects of the problem and methods of approximate numerical computation.  Major results obtained during the grant period (1) extend the set of situations for which nullcontrollability is known  [1], [5] ,

(2)  characterize the optimal control [5], [19], [24] and (3) consider aspects (regularity, dependence, etc.) more-or-less directly related to computation [5], [14]....

Since the computational aspects lead to ill-posed problems — e.g., the minimum norm boundary control from  $u(0) = \omega$  to  $u(T) = \hat{\omega}$  does not depend continuously on the target  $\hat{\omega}$ — it has been relevant to consider general computational aspects of ill-posed problems.  Such problems also rise in connection

with _system_ _identification_, e.g., determining the diffusion coefficient a
in specifying  A  above, or in connection with problems of determining
the input to a diffusion system from observation of the output  [ 3 ], [ 6],
[ 16 ], [ 18] .  The Russian school (Tikhonov...) has been particularly
active in developing computational approaches to ill-posed problems;
these have been of great importance in geophysical contexts.

Entirely independent of the above is a nonlinear diffusion-convec-
tion-reaction system arising in semiconductor theory but, with modifi-
cation, applying quite gnerally to contexts in which the convection is
determined by the electrostatic field which, in turn, is determined by
the (unknown) distributions of various charged species:

$$\dot{u}_k - \nabla \cdot J_k = S_k \quad (k = 1,\dots,K)$$

where the 'currents'  $J_k$  are given by

$$J_k = a_k \nabla u_k + u_k v_k$$

with 'drift velocities'  $v_k$ determined by the field E:

$$V_k = -q_k a_k E \quad (q_k = \text{charge})$$

and $E = -\nabla \psi$ for a potential  $\psi$  given by the usual Poisson equation:

$$-\Delta \psi = N + \sum_k q_k u_k \quad (N=N(\cdot) \text{ given}).$$

Conservation of charge gives  $\sum_k q_k S_k = 0$.

A discussion of the interplay between theoretical and computational
considerations (for the steady state problem, specifically for a semi-
conductor device) appears in [ 12 ].  Major results  [ 10], [30 ], [31 ]
are on existence of solutions with related _a_ _priori_ estimates.

c. Bibliographical list of publications and reports.

Publications:

[1]     Observation and prediction for the heat equation, IV:  patch observa-
bility and controllability.  SIAM J. Contr. Opt. 15 (1977) pp. 64-72.
        (Written before the grant period but final submission with
        credit to the grant for support.)

[2]     Estimation of continuous inputs from imprecise discrete observations.
Proc. 1977 Conf. Inf. Sci. Syst., Johns Hopkins Univ., Baltimore, 1977,
pp. 58-61.

[3]     Parameter and state estimation for a diffusion process.  Proc. 2nd
IFAC Symp. CDPS, IFAC, Düsseldorf, 1978, pp. 239-245.
        (A somewhat more extended form of this material is included
        in [18].)

[4]     Time-invariance of the reachable state for linear control problems.
Proc. 2nd IFAC Symp. CDPS, IFAC, Düsseldorf, 1978, pp. 85-89.
        (Written during the symposium; a revised version appeared
        as [7].)

[5]     Exact boundary controllability for some evolution equations.  SIAM
J. Contr. Opt. 16 (1978), pp. 979-999.

[6]     Some ill-posed problems arising in boundary observation and control
of diffusion equations in Inverse and Improperly Posed Problems in
Differential Equations (edit: G. Anger), Akademie Verlag, Berlin,
1979, pp. 233-247.
        (Presented at conference at Halle, DDR.)

[7]     Time-invariance of the reachable set for linear control problems.
J. Math. Anal. Appl. 72 (1979), pp. 17-20
        (Revised version of [4].)

[8]     Approaches to geophysical inverse problems: a methodological compari-
son.  Part I: a posteriori approach. NASA Tech. Memo 80301, Goddard
Space Flight Ctr., 1979.
        (This appeared as joint with M.J. Munteanu who was to have written
        Part II, giving certain applications; apparently that Part
        will never appear.)

[9]     Nonconvergence results for the application of least squares estima-
tion to ill-posed problems.  J. Opt. Th. Appl. 30 (1980), pp. 535-547.

[10]    Steady state solutions of a nonlinear diffusion-reaction system with
electrostatic convection.  Nonlinear Anal. - TMA 4 (1980), pp. 623-637.
(Note also [12], [30], [31].)

[11]    Recovery of a diffused signal, Séminaires IRIA 1979 (Analyse et
Controle des Systemes), Rocquencourt, 1980, pp. 71-82.
        (Related to the subject of an  Army Workshop on Ballistics at
        Aberdeen.)

[12] A nonlinear elliptic system arising in semiconductor theory, Séminaires IRIA 1979 (Anal. et Cont. des Syst.), Rocquencourt, 1980, pp. 83-95. (Discussion of [10] related to computational methods.)

[13] Boundary control of $u_t = u_{xx} - qu$ and the analyticity of semi-groups with distributed conditions, Séminaires IRIA 1979 (Anal. et Cont. des Syst.), Rocquencourt, 1980, pp. 97-105.

[14] Regularity of optimal boundary controls for parabolic equations, (edit. Bensoussan/Lions) Analysis and Optimization of Systems (Lect. Notes Contr. Inf. Sci. #28), Springer, Berlin, 1980, pp. 536-550. (Presented at IRIA Symp., Versailles, 1980; follows [19].)

[15] The asymptotic growth of solutions of $-\Delta u = \lambda f(u)$ for large $\lambda$. Ind. Univ. Math. J. 30 (1981), pp. 305-311.

To appear (accepted for publication):

[16] Approximation methods in distributed parameter system theory in Ill-Posed Problems: Theory and Practice (edit. M.Z. Nashed), Academic Press, N.Y., 1981 (UMBC MRR 79-18). (Presented at conference, U. Del., 1979).

[17] Convergent approximation schemes for ill-posed problems, Part I: Theory. Control and Cybernetics (1981).

[18] Convergent approximation schemes for ill-posed problems, Part II: applications. Control and Cybernatics (1981). (This and [17] appeared together in report form as UMBC MRR 78-10; versions of parts appeared in [3], [11].)

[19] Regularity of optimal boundary controls for parabolic equations, I: semigroup methods and analyticity. SIAM J. Contr. Opt. (UMBC MRR 80-8).

[20] On the optimal solution of the 'one-armed bandit' adaptive control problem. (with P.Kumar) IEEE Trans. Auto. Contr. (Bellman issue, 1981).

[21] Boundary inequalities for eigenfunctions and boundary control theory. J. Diff. Eqns.

[22] Approximation methods for distributed parameter systems. in Distributed Parameter Control Systems: Theory and Applications (edit. S.G. Tzafestas), Pergamon, Oxford.

Submitted for publication:

[23] L'existence de contrôles optimales pour un problème distribué non-linéaire. to Comptes Rendus Acad. Sci., Paris.

[24] Existence and uniqueness of optimal controls for a semilinear parabolic equation. (with Zhou H-X) to SIAM J. Contr. Opt.

[25] Two results on exact boundary control of parabolic equations. _to_ Appl. Math. Opt.

[26] Existence and regularity of extrema. _to_ J. Math. Anal. Appl.

## Results obtained and now being written (tentative titles):

[27] Identification of the nonlinearity $k(\cdot)$ in $u_t = \nabla \cdot k(u) \nabla u$ from boundary observations.

[28] Existence of solutions for certain semilinear equations.

[29] Coercivity estimates for a class of nonlinear elliptic operators.

[30] A nonlinear parabolic system arising in semiconductor theory. (with G.M. Troianiello).

[31] A nonlinear parabolic system arising in semiconductor theory, II.

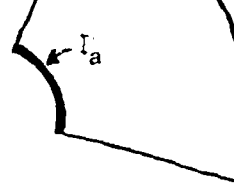[32] A characterization of certain 'nearest points'.

[33] Observation and prediction for a one-phase Stefan problem. (Presented at Symp. on Free Boundary Problems, Montecatini, 1981.)

[34] Partial Differential Equations. (Book; first draft written as notes from a course at Université de Nice 1980-'81.)

## Work in progress:

Certain other lines of investigation have preceeded, during the grant period and since, with some partial results although not to the more 'finished' state of [27] - [33].

d.) Summaries of principal results obtained; abstracts.

[1] The principal result is that the heat equation $u_t = \Delta u$
is exactly nullcontrollable for a region such as
indicated with boundary control restricted to the patch
$\Gamma_a$ and vanishing on $\partial\Omega\backslash\Gamma_a$ . This shows, in particular
that Russell's 'hyperbolic $\Rightarrow$ parabolic' principle is far
from reversible and that star-complementarity is far
from necessary for boundary control and observation of the
heat equation. For a general patch and general region such results remain
unproved but state prediction from such observations is shown to be well-
posed given an a priori bound on initial data.

[2] Let $x(\cdot)$ satisfy the linear equation $\dot{x} = Ax + Bu$ and consider linear
observations of the form $w = \sum_k \lambda_k x_{i(k)}(t_k)$ (finite sum). An imprecise
observation vector $\hat{w}$ is given, each component of which is nominally of
this form (not necessarily linearly independent), and one seeks an opti-
mum (minimum norm) estimate of the input $u(\cdot)$. The equation and obser-
vations define a map: $(\xi, u(\cdot)) \longmapsto \underline{w}$ : (initial data, input) $\longmapsto$ (exact
observation) and a suitable pseudo-inverse is to be constructed relative
to an indefinite inner product which ignores $\xi$ .

[3] Let $u$ be known to satisfy a parabolic equation of the form:
$u_t = \nabla \cdot A\nabla u = qu$ in $(0,T) \times \mathcal{U}$ with $\alpha u + \beta n \cdot A\nabla u = 0$ on
$(0,T) \times \partial\Omega$ and initial data $u_o$. If $u_o$ and the coefficient set
$\underline{\sigma} = (A, q, \alpha, \beta)$ are (partially) unknown, one asks how they may be
determined using additional boundary observations. (The problems are, in
general, ill-posed and the computational considerations are further
explored in [6], [18], [22].)

[4] For general linear control systems, time-invariance of the reachable
set is a purely algebraic corollary of exact nullcontrollobility. The
result is formulated for time-dependent problems: $K_{s,t}$ = (states at
time $t$ reachable from $x$ at time $s$) is independent of $s,x$ for
$s \le r < t$ if every state at $r$ is controllable to $0$ at $t$. (This
generalizes a result of Fattorini and was used by Schmidt in proving the
bang-bang principle for boundary control of the heat equation.)

[5] Russell had shown that a uniform local decay rate implies boundary con-
trollability for the wave equation $u_{tt} = \Delta u$ and that this implies
corresponding controllibility for $u_t = \Delta u$. An abstract version of this
is proved and applied to give controllability for a class of variable
coefficient hyperbolic and parabolic problems. The principal new results
for the parabolic case are: (1) a characterization of minimum norm null-
controls as complementary boundary data of the adjoint equation and (2)
continuous dependence of the optimal control (in $H^s((0,T') \times \partial\Omega$ for
$0 < T' < T$ and weak in $L^2((0,T)\times\partial\Omega)$) on the coefficients of the equa-
tion. (The first of these gives analyticity in $t$ of optimal nullcon-
trols; compare [19]. )

[6] A number of problems are discussed briefly. Principal new results are: (1) an abstract result on computing solutions of linear problems with convex constraints applied to optimal boundary control and (2) uniqueness for some problems of identifying q in $u_t = u_{xx} - qu$.

[7] See [4].

[8] There is a considerable variety of approaches to ill-posed problems (as geophysical inverse problems) which have significant differences in method and interpretation based on varying assumptions regarding, e.g., the nature of the measurement uncertainties. This primarily expository report compares various approaches to a posteriori interpretation of imprecise, inadequate data and gives uncertainty estimates where possible.

[9] Let $Ax = b$ be a linear problem (here presented abstractly). The method of 'least squares' is often applied by writing $x = \sum \alpha_k e_k$ ($\{e_k\}$ an orthonormal basis), taking enough terms $k = 1, \ldots, K$ to permit good approximation and then determining $x_K = \sum_1^K \alpha_k e_k$ to minimize $\|Ax_K - b\|^2$. For an ill-posed problem ($A^{-1}$ unbounded) examples are constructed for which an exact solution exists, exact computation is assumed but $\{x_K\}$ is unbounded as $K$ is increased. Contrary to the situation for well-posed problems, the approximation can get worse as more terms are taken.

[10] A general reaction-diffusion-convection system for charged species is:
$$- \nabla \cdot D_k [\nabla u_k + q_k u_k \nabla \psi] = S_k \qquad (k=1,\ldots,K$$
$$- \Delta \psi = N + \sum q_k u_k$$
where $q_k$ is the charge and $u_k$ the concentration of the k-th species and $\psi$ is the electrostatic potential. Under suitable hypotheses, existence of solutions (with each $u_k$ positive) can be proved.

In particular, this applies to a specific system of semiconductor theory with $D_k$ dependent on $|\nabla \psi|$ and $S_k = -R$ given by the Shockley-Reed-Hall model of recombination.

[11] Let $\varphi = \varphi(t)$ be a thermal transient of interest and suppose one can observe the output $\psi(t) = u(0,t)$ determined by the diffusion equation $u_t = [Du_x]_x$ with $u(1,t) = \varphi(t)$, $u_x(0,t) = 0$ and some initial condition $u = u_0$. For $D = D(x)$ or for $D = D(u)$ analytic in $u$ it is shown that $\varphi$ is uniquely determined by exact knowledge of $\psi$. Some computational algorithms for determining $\varphi$ from $\psi$ are proposed and convergence results stated. (This is related to the problem of determining temperature transients in a gun at firing; there $D = D(x,u)$, after taking polar coordinates.)

[12] The relation is given between the existence result of [10] and an earlier paper (1972) on numerical computation. Since the result uses a substitution $v_k = u_k \exp[q_k \psi]$, an explicit bound on $u_k$ in terms of

the voltage across the device requires estimating the solution of $-\Delta\varphi = \lambda e^{-\varphi}$ as $\lambda$ increases (compare [15]). A calculation is also presented relevant to the question of bifurcation as voltage increases.

[13] It is shown that an infinite horizon optimal control problem has a reformulation as construction of a stabilizing feedback. In this form one has a pure initial value problem giving a semigroup which is shown to be holomorphic.

[14] After a summary of the argument of [19], q.v., asymptotic estimates are obtained as $t \to 0+, T-$ for the $H^s(\partial\Omega)-$ norm of $\varphi(t)$ where $\varphi$ is an optimal Neumann boundary control for the heat equation on $(0,T)\times\Omega$. Similar results are obtained for a semilinear equation. (These estimates are related to the convergence rates obtainable in various methods of numerical computation.)

[15] Generalizing an argument of [12] for $f(u) = e^{-u}$ it is shown that, for $f=e^{-h}$ with $h \nearrow \infty$ and convex, one has max $|u| = \mathcal{O}(f^{-1}(1/\lambda))$ as $\lambda \to \infty$. (For $e^{-u}$ this gives $\log\lambda$ growth. An explicit solution, discovered subsequently, shows that is sharp.)

[16] A comparison is given of numerical methods for computing optimal boundary controls for the heat equation in three contexts: minimum norm controls with no constraint or nonnegativity constraint (semi-constrained: $\mathcal{U}_{ad}$ unbounded) and time-optimal controls with a given bound. The method recommended in general was a projection method (FEM) with regularization. An argument was given to show this projection method converged even without regularization when applied to unconstrained nullcontrol. (Note: The comparison did not include switching-point methods applicable, noting the bang-bang principle, to certain time-optimality problems in one space dimension.)

[17],[18] These were prepared originally in 1978, based on still earlier work, and have been extensively distributed in report form. The principal theoretical results [17] are: (1) a general convergence theorem for approximations $F_k(x) \approx b_k$ to $F(x) = b$ . (The correct notion of convergence $F_k \to F$ is 'graph subconvergence' in $X_w \times Y$ : if $(x_k, y_k)$ in graph $(F_k)$ and $x_k \to x$, $y_k \to y$, then $(x,y)$ in graph $(F)$.) and (2) discussion, with error estimates, of the method of 'generalized interpolation', particularly in the linear case in Hilbert space (approximation $x_K$ to $Ax = b$ given by $\langle x_K, A^*\eta_j \rangle = \langle b, \eta_j \rangle$ for $j = 1,\ldots,K$ with $x_K$ in sp $\{A^*\eta_j\}_1^K$ ).

The applications considered [18] are (1) the inversion of Nemytsky operators(nonlinear back-substitution), (2) recovery of a diffused signal (compare [11], written later) and (3) system identification for systems governed by parabolic and functional differential equations (compare [3], written earlier).

[19]    Let $u_t + Au = 0$ (e.g., $A = -\Delta$), $u|_{\partial\Omega}$ or $u_\nu = \varphi$ with given initial data and suppose $\varphi$ is optimal, minimizing $J = \|\varphi\|_\Sigma^2 + \lambda\|u - \hat{u}\|_Q^2$ $+ \mu\|u(T) - \hat{\omega}\|_\Omega^2$ with the target trajectory $\hat{u}$ analytic in $t$. Then $\varphi$ is analytic in $t$ as an $H^s(\partial\Omega)$ - valued function (s as large as is consistent with the regularity of $\partial\Omega$, $\hat{u}$ and the coefficients of A) on a complex domain including the open real interval $(0,T)$. This is also true for $\mu = \infty$ (exact control to $\hat{\omega}$ assuming this possible) and for infinite horizon (stabilization) problems; compare [5], [13], [14].

[20]    A variety of situations can be modeled by this problem or more complicated variants: one seeks to maximize the expected discounted return associated with a choice between a 'gamble' with known success probability $p_o$ and another with (unknown) probability p. Beginning with an assumed $\beta$ distribution for p, one plays this so long as it provides the greater expectation (with Bayesian update of the distribution). This is thus an 'optimal stopping time' problem with optimal strategy characterized by the 'free boundary' separating the choice regions in parameter space. An approximation is obtained for the free boundary.

[21]    Consider, e.g., the trace $\omega = u|_\Gamma$ for eigenfunctions: $-\Delta u = \lambda u$ in $\Omega$, $u_\nu = 0$ on $\partial\Omega$ with $\|u\|_\Omega = 1$. It is shown that $\|\omega\|_\Gamma$ is bounded below if $\Gamma$ is a large enough part of $\partial\Omega$. For $\omega = u_\nu|_\Gamma$ and $-\Delta u = \lambda u$, $u|_{\partial\Omega} = 0$ one has $\|\omega\|_p \geq \mathcal{O}(\sqrt{\lambda})$. The argument is an application of known results in boundary control theory for $u_{tt} = \Delta u$. Other related results are also obtained.

[22]    This is primarily expository, addressed to an audience of 'engineers'. The discussion of computational approaches to problems of observation, control and identification for parabolic equations focuses on the ill-posed problems which may arise and the implications of this for computation.

[23]    One seeks to minimize $J = \|\varphi\|_\Sigma^2 + \lambda\|u(T) - \hat{\omega}\|_\Omega^2$ where $u_t = \Delta u + f(u)$ on $Q = (0,T) \times \Omega$, $u = \varphi$ on $\Sigma = (0,T) \times \partial\Omega$ and $u(0) = \omega$. Note that, even for $f \equiv 0$, $J(\varphi)$ need not be finite for $\varphi \in L^2(\Sigma)$ and J is not lower semi-continuous. Under suitable hypotheses on f, a technical trick — apparently applicable also to a variety of other problems — shows that if $\{(\varphi_k, u_k)\}$ is a minimizing sequence with $(\varphi_k, u_k) \to (\varphi_*, u_*)$, then $(\varphi_*, u_*)$ minimizes J admissibly (including the consideration of constraints, not necessarily convex with respect to u).

[24]    Let $u_t + Au + f(u) = \varphi$ with $u(0) = 0$, $u|_\Sigma = 0$ and $\varphi$ to minimize $J = \|\varphi\|_\Sigma^2 + \lambda\|u - \hat{u}\|_Q^2 + \mu\|u(T) - \hat{\omega}\|_\Omega^2$. Assume $f' = g$ with $0 \leq g \leq K$. Then (1) optimal controls exist, (2) $\varphi \mapsto u$ is Fréchet differentiable (This is shown here only under dimension limitations: $\Omega \subset \mathbb{R}^m$ with $m < 10$ for $A = -\Delta$, but has since been shown to hold in general.) and (3) if the

data $u, \omega$ is 'small enough' and $\mu^2 < \lambda$ then the optimal $(\varphi_*, u_*)$ is unique and depends continuously on $u, \omega$ and, in a certain sense, on $f$. Boundary control is also considered.

[25]    The first result is exact nullcontrollability by boundary control for certain diffusion equations in one space dimension with time-dependent coefficients; e.g., for $u_t = u_{xx} - q(t,x)u$ on $(0,T) \times (0,1)$ provided

$q$ is analytic in $t$ (at $t = 0$) uniformly in $x \in (0,1)$. The second result is that when a parabolic equation is shown to be exactly boundary nullcontrollable (so there exists $C_T$: initial data $\mapsto$ optimal control to give $u(T) = 0$), then $\log \|C_T\| = \mathcal{O}(1/\sqrt{T})$ as $T \to 0+$; This shows how

much harder one must work to control quickly. Analogous results hold for the dual observation problems.

[26]    The principal result addresses the following difficulty which arises, e.g., in justification of optimality conditions for control problems. Given a minimum $\bar{x}$ of $J: X \to \mathbb{R}$ one may be able to justify differentiability so $J'(\bar{x}) = 0$ if one knows $x \in Y \subset X$ (regularity) while the condition $J'(\bar{x}) = 0$, obtained formally, may imply $x \in Y$ — e.g., if $J'$ has the form $y - K(y)$ with $K: X \to Y$. It is shown that this circular argument may be rigorously justifiable, using Ekeland's approximate variational principle.

[27]    Various nonlinear diffusion processes are governed by equations of the general form $u_t = \nabla \cdot k(u) \nabla u$. (Examples of such problems are seepage of water or oil underground and heat conduction with a temperature-dependent coefficient.) To identify the particular nonlinearity $k(\cdot)$ from boundary observations is, in general, an ill-posed problem. For $0 < k$ analytic and Dirichlet data $\varphi = \varphi(x)$ (smooth but not constant), observe the resulting flux $\psi = k(u) \nabla u \cdot n$ on $(0,T) \times \partial\Omega$. It is shown that the pair $(\varphi, \psi)$ uniquely determines $k(\cdot)$. (It is anticipated that a similar result holds if the flux is specified and the trace observed. Some specific numerical procedures may be included as well.)

[28]    If $f: \mathbb{R} \to \mathbb{R}$ is bounded above as $r \to \infty$ and below as $r \to -\infty$ and if $A$ provides a suitable maximum principle (e.g., $A = -\Delta$ with bounded Dirichlet data), then $Au = f(u)$ has a solution.

[29]    Consider $A: u \longmapsto \nabla \cdot a(\cdot, |\nabla u|) \nabla u$ with $ra(\cdot, r)$ having suitable growth properties. Coercivity estimates are obtained giving existence, uniqueness and continuous dependence results for solutions of related elliptic and parabolic problems. An example would be $a(\cdot, r) = r^\alpha / [\omega(\cdot) + r^\alpha]$ with $\alpha > 0$, $0 \le \omega \in L^1$. An application given is to generation of periodic eddy currents in an inhomogeneous nonlinearly magnetic material.

[30], [31]  The system governing diffusion and convection of hole and electron concentrations, with Shockley-Reed-Hall recombination, in a semiconductor device (compare [10] for the steady state problem) is:

$$u_t - \nabla \cdot a[\nabla u + u \nabla \psi] = -R,$$

$$v_t - \nabla \cdot b[\nabla v - v \nabla \psi] = -R,$$

$$- \Delta \psi = N + u - v,$$

$$R = [uv - n^2]/\tau, \quad \tau = \tau(\cdot, u, v).$$

Existence results are obtained for positive solutions of this system. The argument of [30] is for $a, b, \tau$ constant, $n^2 = 1$. The argument of [31] considers $a, b$ functions of $x$, $|\nabla \psi|$ and $\tau = \tau(\cdot, u, v, |\nabla \psi|)$, although under somewhat different hypotheses than in [10]. It is anticipated that this can be coupled with heat conduction and permits $n = n(x)$, not constant.

[32]   For $C$ closed and convex in a Hilbert space $H$ and $S$ a subspace one considers when the nearest point to $0$ in $C \cap S$ is also the nearest point in $C$ to some $x \in S^\perp$. This is related to a representation theorem for certain optimal control problems with constraints as, e.g., nonnegative control.

[33]   The determination of the state for a Stefan problem from observation of the motion of the free boundary is a well-posed problem.

[34]   A book in three main parts:  (1) Elliptic Equations, (2) Parabolic Equations, (3) Observation and Control of Parabolic Equations.  The emphasis in (1) is on variational methods with applicability to nonlinear problems.  The emphasis in (2) is on semigroup methods and semilinear equations with inhomogeneous boundary conditions.  Part (3) treats various topics, concentrating primarily on boundary observation and control along the lines of  [5], [13], [19], [23], [24], etc.